Chaos, Solitons and Fractals 42 (2009) 571–576

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos

A new analytical approximation to the Duffing-harmonic oscillator

M. Fesanghary [a], T. Pirbodaghi [b], M. Asghari [b,*], H. Sojoudi [a]

[a] *Department of Mechanical Engineering, Louisiana State University, Baton Rouge, LA 70803, USA*
[b] *School of Mechanical Engineering, Sharif University of Technology, Azadi Ave., 11365-9567 Tehran, Iran*

ARTICLE INFO

Article history:
Accepted 21 January 2009

Communicated by Prof. Hi-Huan He

ABSTRACT

In this paper, a novel analytical approximation to the nonlinear Duffing-harmonic oscillator is presented. The variational iteration method (VIM) is used to obtain some accurate analytical results for frequency. The accuracy of the results is excellent in the whole range of oscillation amplitude variations.

1. Introduction

Consider a nonlinear oscillator modeled by the following governing nonlinear differential equation [1]:

$$\frac{d^2u}{dt^2}+\frac{u^3}{1+u^2}=0,\quad u(0)=A,\quad \frac{du}{dt}(0)=0 \tag{1}$$

For small values of parameter u, the governing Eq. (1) is that of a Duffing-type nonlinear oscillator, i.e., $\frac{d^2u}{dt^2}+u^3\cong 0$, while for large values of u the equation approximates that of a linear harmonic oscillator, i.e., $\frac{d^2u}{dt^2}+u\cong 0$. Hence, Eq. (1) is called the Duffing-harmonic oscillator [1].

Due to the highly nonlinearity of differential Eq. (1), no exact analytical solution has been presented for it in the literature. However, researchers have been concentrated on approximate analytical techniques for this equation. There are many approaches for approximating solutions of the Duffing-harmonic oscillator. The most common method is the harmonic balance method [2]. By applying the method of harmonic balance (HB) [3], the angular frequency is obtained as [1]

$$\omega^2=\frac{3}{4}A^2\left(1+\frac{3}{4}A^2\right)^{-1} \tag{2}$$

Using the energy balance method, Oziş and Yıldırım [4] obtained the angular frequency in the following form

$$\omega^2=1-\frac{2}{A^2}\ln\left(\frac{1+A^2}{1+\frac{A^2}{2}}\right) \tag{3}$$

Assuming a single-term solution and applying the Ritz procedure [5], Tiwari et al. [6] obtained an approximate frequency in the following form

$$\omega^2=1+\frac{2}{A^2}\left(\frac{1}{\sqrt{1+A^2}}-1\right) \tag{4}$$

* Corresponding author. Tel.: +98 21 66165571.
E-mail address: asghari@sharif.edu (M. Asghari).

0960-0779/$ - see front matter
doi:10.1016/j.chaos.2009.01.024

Using the homotopy perturbation method (HPM), He [7] obtained the angular frequency of Eq. (1). The first-order approximation of HPM gives a frequency–amplitude relation identical with those of HB solution, presented in Eq. (2). It is noted that the exact frequency of Eq. (1) is obtained if one can integrate the following equation [8]

$$\omega_e = \frac{\pi}{2}\left(\int_0^{\pi/2} \frac{A^2\cos^2\theta d\theta}{\sqrt{A^2\cos^2\theta + \ln\left(1 - A^2\cos^2\theta/1 + A^2\right)}}\right)^{-1}. \tag{5}$$

To obtain an accurate analytical solution for frequency–amplitude relation of the Duffing-harmonic oscillator, this paper employs He's variational iteration method [9] which is a powerful mathematical tool for various kinds of nonlinear problems.

2. Variational iteration method

The principles of the VIM and its applicability for various kinds of nonlinear differential equations are given in [9–12]. The VIM can lead to convenient approximate solutions to all kinds of nonlinear equations with simple solution procedure. To illustrate the basic idea of the method, we consider the following general nonlinear system:

$$L[u(t)] + N[u(t)] = g(t), \tag{6}$$

where $\boldsymbol{L}$ is a linear operator, $\boldsymbol{N}$ is a nonlinear operator and g(t) is a given continuous function. The basic character of the method is to construct a correction functional for the system as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)[\mathrm{L}u_n(\tau) + \mathrm{N}\tilde{u}_n(\tau) - g(\tau)]d\tau, \tag{7}$$

where λ is a Lagrange multiplier, which can be optimally determined via variational theory. Also, u_n is the nth approximate solution and $\tilde{u}_n$ represents a restricted variation, i.e., $\delta\tilde{u}_n = 0$.

3. Implementation of the VIM

The governing Eq. (1) can be rewritten in the form

$$(1 + u^2)\frac{d^2u}{dt^2} + u^3 = 0. \tag{8}$$

Assuming

$$\Omega^2 = \psi, \tag{9}$$

one can write

$$\frac{d^2u}{dt^2} + \Omega^2 u = \mathrm{F}[u], \tag{10}$$

where

$$\mathrm{F}[u] = -u^2\frac{d^2u}{dt^2} - u^3 + \psi u. \tag{11}$$

The correction functional can be constructed in the following form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\left\{\frac{d^2u_n(\tau)}{d\tau^2} + \Omega^2 u_n(\tau) - \widetilde{\mathrm{F}}[u_n(\tau)]\right\}d\tau. \tag{12}$$

$\widetilde{\mathrm{F}}[u_n]$ is considered as a restricted variation, i.e., $\delta\widetilde{\mathrm{F}}[u_n] = 0$. Calculating the variation of Eq. (12) and noting that $\delta\widetilde{\mathrm{F}}[u_n] = 0$, the following stationary conditions is obtained:

$$\begin{cases} \frac{d^2\lambda}{d\tau^2}(\tau) + \Omega^2\lambda(\tau) = 0, \\ \lambda(t) = 0, \\ 1 - \frac{d\lambda}{d\tau}(t) = 0. \end{cases} \tag{13}$$

The Lagrange multiplier, therefore, can be easily identified as

$$\lambda(\tau) = \frac{1}{\Omega}\sin\Omega(\tau - t). \tag{14}$$

On the other hand, by taking into consideration the relation

$$\int_0^t \sin\Omega(\tau - t)\left\{\frac{d^2u_n(\tau)}{d\tau^2} + \Omega^2 u_n(\tau)\right\}d\tau = -\Omega u_n(t) + \Omega u_n(0)\cos\Omega t + \frac{du_n}{dt}(0)\sin\Omega t, \tag{15}$$

Eq. (12) can be rewritten as

$$u_{n+z1}(t) = u_n(0)\cos\Omega t + \frac{du_n}{dt}(0)\frac{\sin\Omega t}{\Omega} - \frac{1}{\Omega}\int_0^t \sin\Omega(\tau - t)\{\mathrm{F}[u_n(\tau)]\}d\tau. \tag{16}$$

Considering the initial conditions $u(0) = A$ and $\frac{du}{dt}(0) = 0$, the correction functional can be further simplified as follows:

$$u_{n+1}(t) = A\cos\Omega t - \frac{1}{\Omega}\int_0^t \sin\Omega(\tau - t)\{\mathrm{F}[V_n(\tau)]\}d\tau. \tag{17}$$

As initial guess, $u_0(t)$ can be considered as follows:

$$u_0(t) = A\cos\Omega t. \tag{18}$$

Expanding $\mathrm{F}[u_0(t)]$, we have:

$$\mathrm{F}[u_0(t)] = \left[\frac{3A^3}{4}(1-\Omega^2) - \psi A\right]\cos\Omega t + \left[\frac{A^3}{4}(1-\Omega^2)\right]\cos 3\Omega t. \tag{19}$$

By taking into consideration the relation

$$\frac{1}{\Omega}\int_0^t \sin\Omega(\tau - t)\{\cos n\Omega\tau\}d\tau = \begin{cases} \frac{\cos n\Omega t - \cos\Omega t}{\Omega^2(n^2-1)}, & n \neq 1, \\ \frac{t\sin\Omega t}{-2\Omega}, & n = 1. \end{cases} \tag{20}$$

To avoid secular terms in the next iterations, the coefficient of the $\cos\Omega t$ in $\mathrm{F}[u_n(t)]$ should be vanished. It follows that

$$\psi = \frac{3A^2(1-\Omega^2)}{4}. \tag{21}$$

From Eq. (9) the first approximation of the frequency (i.e., Ω_1) is obtained as follows:

$$\Omega^2 = \Omega_0^2 = \frac{3}{4}A^2\left(1 + \frac{3}{4}A^2\right)^{-1}. \tag{22}$$

This frequency is the same as those obtained using HB and HPM method. From Eqs. (17) and (20) for $n = 1$, first-order approximate solution is obtained as:

$$u_1(t) = A\cos\Omega t + \frac{A^3(1-\Omega^2)(\cos 3\Omega t - \cos\Omega t)}{32\Omega^2}, \tag{23}$$

where frequency Ω is listed in Eq. (20), and therefore:

$$\begin{aligned}\mathrm{F}[u_1(t)] = {} & \left[\frac{3B^3}{4}(1-\Omega^2) + \frac{B^2C}{4}(3-11\Omega^2) + \frac{BC^2}{4}(6-19\Omega^2) - \psi B\right]\cos\Omega t \\ & + \left[\frac{B^3}{4}(1-\Omega^2) + \frac{3C^3}{4}(1-9\Omega^2) + \frac{B^2C}{2}(3-11\Omega^2) - \psi C\right]\cos 3\Omega t \\ & + \left[\frac{BC}{4}(3C - 19C\Omega^2 + 3B - 11B\Omega^2)\right]\cos 5\Omega t + \left[\frac{BC^2}{4}(1-19\Omega^2)\right]\cos 7\Omega t + \left[\frac{C^3}{4}(1-9\Omega^2)\right]\cos 9\Omega t,\end{aligned} \tag{24}$$

where

$$\begin{aligned} B &= A\left[1 + \tfrac{A^2(1-\Omega^2)}{32\Omega^2}\right], \\ C &= \tfrac{A^3(1-\Omega^2)}{32\Omega^2}. \end{aligned} \tag{25}$$

Avoiding the secular term in the next iteration, needs:

$$\psi = \frac{A^2(1-\Omega^2)}{2048\Omega^4}\left[(15A^4 - 80A^2 + 1536)\Omega^4 - (18A^4 + 48A^2)\Omega^2 + 3A^4\right]. \tag{26}$$

Substitution of Eq. (26) into Eq. (9) results in the second approximate of the frequency (i.e., Ω_2) as:

$$\Omega^2 = \frac{A^2(1-\Omega^2)}{2048\Omega^4}\left[(15A^4 - 80A^2 + 1536)\Omega^4 - (18A^4 + 48A^2)\Omega^2 + 3A^4\right]. \tag{27}$$

From Eqs. (17) and (20) the second-order approximate solution is given by:

$$\begin{aligned} u_2(t) = A\cos\Omega t &+ \left[\frac{B^3}{4}(1-\Omega^2) + \frac{3C^3}{4}(1-9\Omega^2) + \frac{B^2C}{2}(3-11\Omega^2) - \psi C\right]\frac{(\cos 3\Omega t - \cos\Omega t)}{8\Omega^2} \\ &+ \left[\frac{BC}{4}(3C - 19C\Omega^2 + 3B - 11B\Omega^2)\right]\frac{(\cos 5\Omega t - \cos\Omega t)}{24\Omega^2} + \left[\frac{BC^2}{4}(1-19\Omega^2)\right]\frac{(\cos 7\Omega t - \cos\Omega t)}{48\Omega^2} \\ &+ \left[\frac{C^3}{4}(1-9\Omega^2)\right]\frac{(\cos 9\Omega t - \cos\Omega t)}{80\Omega^2}. \end{aligned} \tag{28}$$

In the same way by calculating ψ from the coefficient of cos Ωt in F[$u_2(t)$], the third approximate of the frequency Ω_3 can be obtained as:

$$\Omega^2 = \frac{A^2(1-\Omega^2)(D_1 + D_2)}{\Omega^{16} D_3}, \tag{29}$$

where D_1, D_2 and D_3 are as presented in the following

$$\begin{aligned} D_1 = 10^{-7}\Big\{ &(-0.016A^{24} - 0.117A^{22})\Omega^2 + (4.53A^{22} + 0.281A^{24} + 12.9A^{20})\Omega^4 + (72.2A^{22} - 2.67A^{24} - 535A^{18} \\ &- 459A^{20})\Omega^6 + (6700A^{20} + 617A^{22} + 18700A^{18} + 1450A^{16}15.1A^{24})\Omega^8\Big\} + 10^{-4}\{(-3.11A^{22} - 15.6A^{16} + 255A^{14} \\ &- 5.33A^{24} - 521A^{20})\Omega^{10} + (9.64A^{22} + 235A^{20} - 5100A^{12} + 1960A^{18} + 0.123A^{24})\Omega^{12} + (-638A^{20} - 0.191A^{24} \\ &- 19.1A^{22} - 8130A^{18} - 230A^{10} - 28000A^{16} + 31900A^{14} + 134000A^{12})\Omega^{14}\Big\}, \end{aligned}$$

$$\begin{aligned} D_2 = &(-71.6A^{10} + 0.107A^{20} + 118A^8 + 11.9A^{16} + 1.93A^{18} - 119A^{12} - 0.76A^{14} + 0.003A^{22})\Omega^{16} + (-0.002A^{22} - 1290A^8 \\ &- 25.7A^{16} - 426A^{12} - 1580A^6 + 984A^{10} - 0.112A^{20} - 46.1A^{14} - 2.69A^{18})\Omega^{18} + (20600A^6 + 1280A^8 + 28.9A^{16} \\ &+ 118A^{14} - 3930A^{10} - 555A^{12} + 2.16A^{18} + 0.072A^{20} + 0.001A^{22})\Omega^{20} + (7240A^8 - 0.026A^{20} - 46000A^6 + 247A^{12} \\ &- 0.0003A^{22} - 108A^{14} + 16.3A^{16} + 5200A^{10} - .939A^{18} - 99800A^4 + 18400A^2)\Omega^{22}, \end{aligned}$$

$$\begin{aligned} D_3 = &(6A^8) - (780A^6 + 67A^8)\Omega^2 + (168 * A^8 + 6820A^6 + 3840A^4)\Omega^4 - (153A^8 + 11300A^6 + 79360A^4 - 122880A^2)\Omega^6 \\ &+ (49A^8 + 5260A^6 + 75520A^4 - 122880A^2 - 3932160)\Omega^8. \end{aligned}$$

4. Results and discussions

To show the accuracy of the proposed method, the approximate frequencies computed by Eqs. 22,27 and 29 are compared with those obtained by other researchers. Table 1 shows the result for various amplitudes. From Table 1 it can be observed that Eqs. (27) and (29) yield excellent approximate frequencies for both small and large amplitudes.

Figs. 1–3 have been presented to compare the obtained analytical results of this paper from variational iteration method with numerical results obtained by Runge–Kutta method. From these figures, it is obvious that the results of this paper are nearly identical with those given by numerical method.

Table 1
Comparison of various approximate angular frequencies.

Method	A				
	0.01	0.1	1	10	100
Mickens [1], He [7] (Eq. (2))	0.00866	0.08628	0.65465	0.99340	0.99993
Oziş and Yıldırım [4] (Eq. (3))	0.00806	0.08627	0.65164	0.99314	0.99993
Tiwari et al. [5] (Eq. (4))	0.00837	0.08627	0.64359	0.99095	0.99990
Present study					
Ω_1 (Eq. (22))	0.00866	0.08628	0.65465	0.99340	0.99993
Ω_2 (Eq. (27))	0.00849	0.08452	0.62636	0.99999	1.00000
Ω_3 (Eq. (29))	0.00847	0.08439	0.64154	0.99093	0.99992
Exact solution [8]	0.00847	0.08439	0.63678	0.99092	0.99990

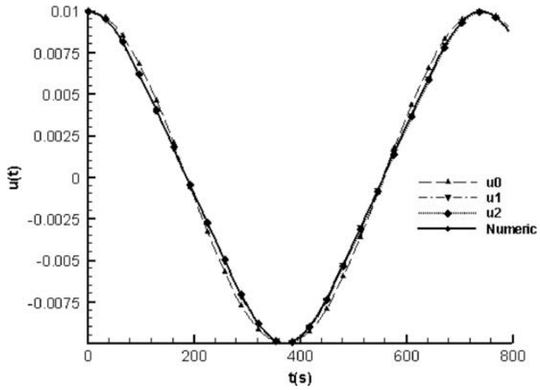

Fig. 1. Comparison of approximate solutions with exact solution for $A = 0.01$.

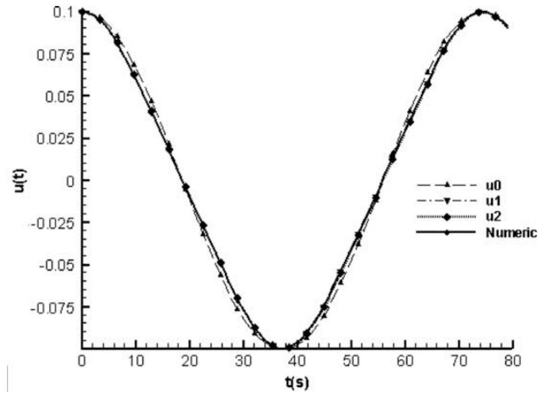

Fig. 2. Comparison of approximate solutions with exact solution for $A = 0.1$.

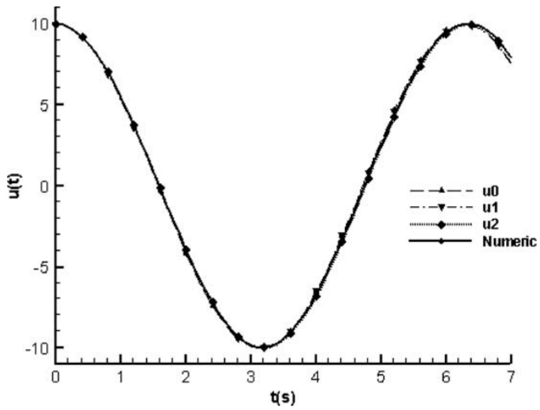

Fig. 3. Comparison of approximate solutions with exact solution for $A = 10$.

5. Conclusions

In this study, analytical solutions for frequency–amplitude relations of the Duffing-harmonic oscillator were presented using variational iteration method. The approximate analytical frequencies are valid for the whole range of oscillation amplitudes. The performance of the method was compared with other approximate analytical methods. Results reveal that this method can be considered as viable alternative for conventional methods to solve highly nonlinear oscillatory systems.

References

[1] Mickens RE. Mathematical and numerical study of the Duffing-harmonic oscillator. J Sound Vib 2001;244:563–7.
[2] Mickens RE. A generalization of the method of harmonic balance. J Sound Vib 1986;111:515–8.
[3] Mickens RE. Oscillations in planar dynamic systems. Singapore: Scientific; 1966.
[4] Oziş T, Yıldırım A. Determination of the frequency–amplitude relation for a Duffing-harmonic oscillator by the energy balance method. Comput Math Appl 2007;54:1184–7.
[5] Abramson HN. Nonlinear vibration. In: Harris CM, editor. Shock and vibration handbook. New York: McGraw-Hill; 1988.
[6] Tiwari SB, Rao BN, Swamy NS, Sai KS, Nataraja HR. Analytical study on a Duffing-harmonic oscillator. J Sound Vib 2005;285:1217–22.
[7] He JH. Non-perturbative methods for strongly nonlinear problems. Dissertation. de-Verlag im Internet GmbH, 2006.
[8] Lim CW, Wu BS. A new analytical approach to the Duffing-harmonic oscillator. Phys Lett A 2003;311:365–73.
[9] He JH. A new approach to nonlinear partial differential equations. Commun Nonlinear Sci Numer Simul 1997;2(4):230–5.
[10] Batiha B, Noorani MSM, Hashim I. Application of variational iteration method to the generalized Burgers–Huxley equation. Chaos, Solitons and Fractals 2008;36:660–3.
[11] Javidi M, Jalilian Y. Exact solitary wave solution of Boussinesq equation by VIM. Chaos, Solitons and Fractals 2008;36:1256–60.
[12] Inc M. Numerical simulation of KdV and mKdV equations with initial conditions by the variational iteration method. Chaos, Solitons and Fractals 2007;34:1075–81.